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The optimal thickness of a wall with convection on one side

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Abstract—This paper documents the conjugate heat transfer through a wall with nonuniform thickness, which is lined on one side by a boundary layer. In the first part, variational calculus shows that the total heat transfer rate is minimized when the wall thickness decreases in an optimal manner in the direction of flow. The reductions in total heat transfer rate are significant when the Biot number is smaller than 1. In the second part of the study, the complete problem of a laminar forced convection boundary layer coupled with conduction through a variable-thickness wall is solved numerically. Means for calculating the total heat transfer rate are reported graphically. It was again found that the total heat transfer rate decreases when the wall profile is tapered so that the wall thickness decreases in the direction of flow.

THE PROBLEM

AN IMPORTANT characteristic of many convection heat transfer configurations is that the heat transfer coefficient varies substantially in the flow direction x. For example, in a forced-convection laminar boundary layer over a flat wall h decreases as $x^{-1/2}$, while in a natural-convection laminar boundary layer h decreases as $x^{-1/4}$. When the wall that is swept by the convective flow has a finite thickness and thermal conductivity, its thermal resistance is added to the resistance of the boundary layer. Intuitively, it seems that a larger wall thickness will have its greatest insulation effect in that wall section over which the convection heat transfer coefficient is large.

Consider the wall with variable thickness $\delta(x)$ shown in Fig. 1. On one side of the wall, the heat transfer coefficient is large enough so that the temperature of that surface is uniform, T_0. The other side is exposed to a flow of different temperature $(T_0+\Delta T)$, across a heat transfer coefficient whose variation along the wall is known, $h(x)$. The wall length L is also specified.

The local heat flux driven by the overall constant temperature difference ΔT is

$$q'' = \frac{\Delta T}{\dfrac{1}{h}+\dfrac{\delta}{k_w}}. \tag{1}$$

By integrating this over the entire length L we obtain the total heat transfer rate q', expressed per unit length in the direction perpendicular to the plane of Fig. 1

$$q' = \int_0^L \frac{\Delta T\,dx}{\dfrac{1}{h}+\dfrac{\delta}{k_w}}. \tag{2}$$

Of interest is the optimal wall thickness distribution $\delta(x)$ for which the heat transfer integral q' is minimum, while the volume of wall material is fixed. The volume (per unit length) constraint can be written as

$$\int_0^L \delta\,dx = \bar{\delta}L \quad \text{(constant)} \tag{3}$$

in which $\bar{\delta}$ (fixed) is the L-averaged thickness of the wall.

SOLUTION BY VARIATIONAL CALCULUS

The minimization of the integral (2) subject to the integral constraint (3) is equivalent to the minimization of the aggregate integral

$$\Phi = \int_0^L \left(\frac{\Delta T}{\dfrac{1}{h}+\dfrac{\delta(x)}{k_w}} + \lambda\delta(x) \right) dx = \int_0^L F\,dx \tag{4}$$

subject to no constraints (see, for example, Bejan [1]). The factor λ in the integrand is a Lagrange multiplier. The optimal thickness can be determined by solving the Euler equation

$$\frac{\partial F}{\partial \delta} - \frac{d}{dx}\left[\frac{\partial F}{\partial(d\delta/dx)}\right] = 0 \tag{5}$$

in which F is shorthand for the integrand of the Φ

NOMENCLATURE

b	dimensionless taper parameter, equation (27)
Bi	Biot number, equation (12), $\bar{\delta} h_L/k_w$
f	function provided by Blasius' solution, $f(\eta)$
F	integrand of integral (4)
$h(x)$	heat transfer coefficient
h_L	value of h at $x = L$
J	dimensionless group, equation (25)
$\bar{J}$	physical constraint group, equation (29)
k	fluid thermal conductivity
k_w	wall thermal conductivity
L	wall length
n	exponent, equation (9)
Nu_x	local Nusselt number, equation (31)
Pr	Prandtl number
q'	total heat transfer rate per unit length, equation (2)
q''	local heat flux
Re_x	Reynolds number, $U_\infty x/\nu$
T_0	reference temperature
ΔT	temperature difference (constant)
u	longitudinal velocity component
U_∞	free stream velocity
x	downstream coordinate, Fig. 1
x_c	distance to region of zero thickness, Fig. 4
y	transversal coordinate, Fig. 1.

Greek symbols

$\delta(x)$	wall thickness
$\bar{\delta}$	L-averaged wall thickness, equation (3)
η	similarity variable, equation (22)
θ	dimensionless fluid temperature, equation (22)
λ	Lagrange multiplier
ν	kinematic viscosity
ξ	dimensionless coordinate, x/L
ξ_c	dimensionless distance, x_c/L
Φ	aggregate integral (4).

Subscripts

c	property of wall with constant thickness, $\delta = \bar{\delta}$
max	maximum
min	minimum
opt	optimal.

integral (4). In this problem, equation (5) reduces to $\partial F/\partial \delta = 0$, and the resulting expression for the optimal thickness is

$$\delta_{\text{opt}}(x) = \lambda^{-1/2}(k_w \Delta T)^{1/2} - \frac{k_w}{h(x)}. \tag{6}$$

The Lagrange multiplier is evaluated by substituting (6) into the volume constraint (3)

$$\lambda^{-1/2}(k_w \Delta T)^{1/2} = \bar{\delta} + \frac{1}{L}\int_0^L \frac{k_w}{h(x)}\,\mathrm{d}x \tag{7}$$

so that, in the end, the optimal distribution of wall thickness reads

$$\delta_{\text{opt}}(x) = \bar{\delta} + \frac{1}{L}\int_0^L \frac{k_w}{h(x)}\,\mathrm{d}x - \frac{k_w}{h(x)}. \tag{8}$$

RESULTS

In order to see how the use of the optimal thickness has the effect of decreasing the overall heat transfer rate per unit length q', let us assume that the x-dependence of h is of the boundary layer type

$$h = h_L \xi^{-n}, \quad \text{where } \xi = \frac{x}{L}. \tag{9}$$

For example, the exponent n is 1/2 in laminar forced

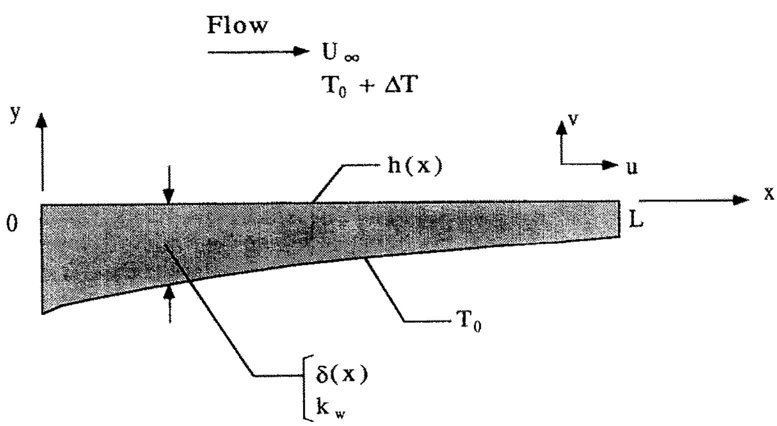

FIG. 1. Wall with variable thickness and convection on one side.

convection and 1/4 in laminar natural convection. The h_L factor is the lowest value of the heat transfer coefficient h, namely the value at the downstream end $x = L$.

If we now combine equations (8) and (9), and substitute the result in the q' integral (2), we obtain the optimal thickness

$$\frac{\delta_{\mathrm{opt}}}{\bar{\delta}} = 1 + \frac{1}{Bi}\left(\frac{1}{n+1} - \xi^n\right) \tag{10}$$

and, cf. equation (2), the minimum heat transfer rate

$$q'_{\mathrm{min}} = \frac{h_L L \Delta T}{Bi + (1+n)^{-1}}. \tag{11}$$

The Biot number Bi is based on h_L and the L-averaged wall thickness

$$Bi = \frac{h_L \bar{\delta}}{k_w}. \tag{12}$$

The physical fact that $\delta_{\mathrm{opt}}/\bar{\delta} > 0$ places a constraint on the Biot number range in which the solution (10) is valid

$$Bi > \xi^n - \frac{1}{n+1}. \tag{13}$$

The most stringent constraint of this type corresponds to $\xi = 1$ (or $x = L$); therefore the allowable Bi range is

$$Bi \geqslant \frac{n}{n+1}. \tag{14}$$

It is worth noting that when $Bi = n/(n+1)$ the optimal thickness drops to zero at the downstream end of the wall, $x = L$.

The minimum heat transfer rate can be compared with the heat transfer rate through a constant-thickness wall that has the same volume, cf. equation (2)

$$q'_c = \int_0^L \frac{\Delta T \, \mathrm{d}x}{\dfrac{1}{h_L \xi^{-n}} + \dfrac{\bar{\delta}}{k_w}} = h_L L \Delta T \int_0^1 \frac{\mathrm{d}\xi}{\xi^n + Bi}. \tag{15}$$

The relative merit of the variable-thickness design is indicated by the ratio

$$\frac{q'_c}{q'_{\mathrm{min}}} = \left(Bi + \frac{1}{n+1}\right) \int_0^1 \frac{\mathrm{d}\xi}{\xi^n + Bi} \tag{16}$$

which clearly approaches 1 when the x-dependence of h fades away ($n \to 0$). This ratio (equation (16)) has been calculated and plotted in Fig. 2 (the solid curves) as a function of n and Bi. Each curve is terminated by a circle at the lowest Bi value allowed by criterion (14). An example of optimal wall thickness distribution (equation (10)) is presented in Fig. 3.

This solution has been extended to Biot numbers lower than the threshold (equation (14)) by considering a wall whose inventory of material is distributed only over the upstream portion of length x_c (Fig. 4). The downstream portion ($L - x_c$) is backed by a wall of zero thickness. The analysis contained between equations (2) and (10) can be repeated while holding x_c constant (in place of L). The resulting expression for the optimal thickness subject to the volume constraint (3) is

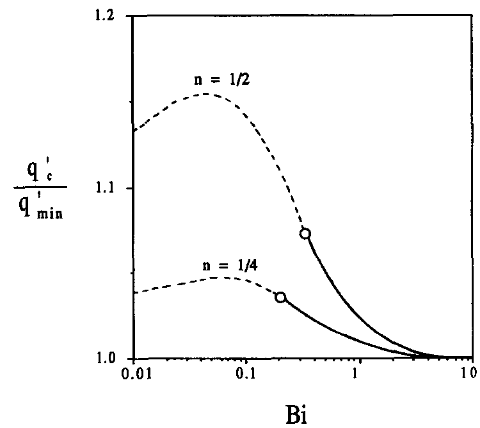

FIG. 2. The total heat transfer rate through the constant-thickness wall, divided by the heat transfer rate through the corresponding wall with optimal thickness.

$$\frac{\delta_{\mathrm{opt}}}{\bar{\delta}} = \frac{1}{\xi_c} + \frac{1}{Bi}\left(\frac{\xi_c^n}{n+1} - \xi^n\right) \tag{17}$$

in which $\xi_c = x_c/L$ and, as before, $\xi = x/L$. Two examples of this optimal-thickness function are given in Fig. 5. By setting $\delta_{\mathrm{opt}} = 0$ at $x = x_c$, we obtain the relationship between ξ_c and Bi

$$Bi = \frac{n}{1+n} \xi_c^{1+n}. \tag{18}$$

This equation shows that ξ_c decreases monotonically as Bi decreases, i.e. that the fixed volume of wall material is positioned closer to the leading edge of the boundary layer as Bi becomes small. Also worth noting is that equation (18) agrees with the inequality (14) when $\xi_c = 1$, and that the geometric range $0 < x_c < L$ of Fig. 4 corresponds to $Bi < n/(n+1)$.

The total heat transfer rate through the wall of Fig. 4 is easily evaluated using equation (2) with the δ_{opt}

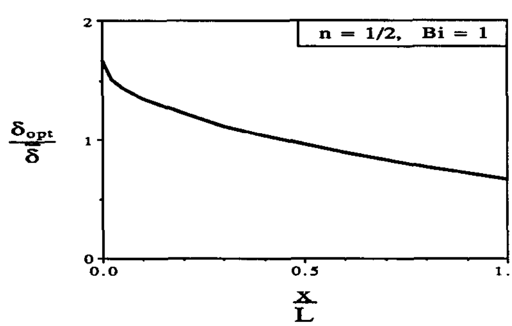

FIG. 3. The optimal variation of wall thickness according to equation (10).

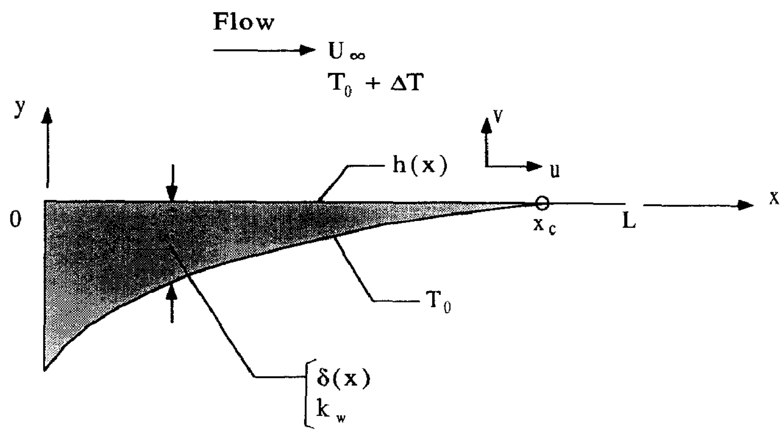

FIG. 4. Wall with variable thickness over the leading section $(0-x_c)$, and zero thickness over the trailing section (x_c-L).

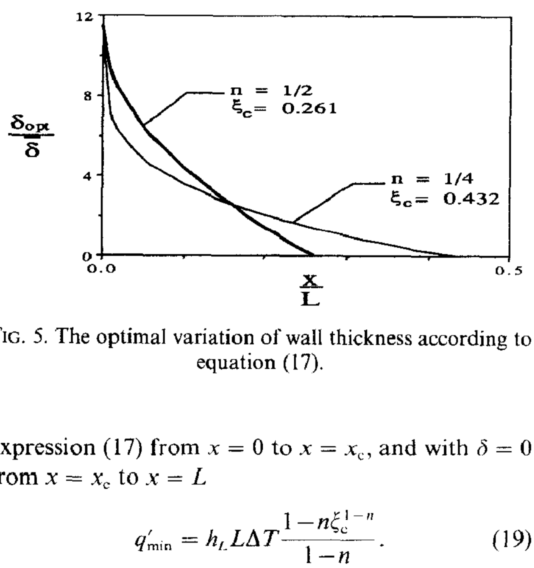

FIG. 5. The optimal variation of wall thickness according to equation (17).

expression (17) from $x = 0$ to $x = x_c$, and with $\delta = 0$ from $x = x_c$ to $x = L$

$$q'_{\min} = h_L L \Delta T \frac{1-n\xi_c^{1-n}}{1-n}. \tag{19}$$

The dimensionless figure of merit constructed in equation (16) assumes the new form

$$\frac{q'_c}{q'_{\min}} = \frac{1-n}{1-n\xi_c^{1-n}} \int_0^1 \frac{d\xi}{\xi^n + Bi} \tag{20}$$

for which the function $\xi_c(Bi)$ is provided implicitly by equation (18). This dimensionless ratio has been plotted with a dashed line in Fig. 2, showing that it reaches a maximum ($q'_{\min}$ reaches a minimum) at a certain Biot number that depends on the exponent n assumed in equation (9). This feature is detailed numerically in Table 1.

Table 1. The location of the maximum of the ratio $q'/q'_{\min}$ reported in Fig. 2

n	$Bi_{\max}$	$(q'_c/q'_{\min})_{\max}$
1/4	0.066	1.047
1/3	0.065	1.077
1/2	0.045	1.155

THE POHLHAUSEN PROBLEM FOR A WALL WITH VARIABLE THICKNESS

The preceding conclusions were made possible by the simplification adopted in equation (9), in which the x-dependence of the heat transfer coefficient was assumed to be known. In reality, both the heat transfer coefficient and the fluid-side temperature of the wall are consequences of the interaction between convection in the fluid and conduction in the wall. In this section we discard equation (9), and focus instead on the conjugate convection–conduction heat transfer across the temperature difference ΔT maintained between the free stream and the underside of the wall (Fig. 1).

In terms of the usual laminar boundary layer notation, the heat transfer in the fluid is governed by the energy equation (see, for example, Kays and Crawford [2], and Bejan [3])

$$\theta'' + \tfrac{1}{2} Pr f \theta' = 0 \tag{21}$$

where $(\ )' = d(\ )/d\eta$, and

$$\theta(\eta) = \frac{T-T_0}{\Delta T}, \quad \eta = \frac{y}{x} Re_x^{1/2}. \tag{22}$$

The function $f(\eta)$ is known from the Blasius solution for the velocity boundary layer, $u/U_\infty = f'(\eta)$. The Reynolds number is defined as $Re_x = U_\infty x/\nu$.

The two boundary conditions on $\theta(\eta)$ are

$$\theta = 1 \quad \text{as} \quad \eta \to \infty \ (y \to \infty) \tag{23}$$

$$\theta = J\frac{\partial \theta}{\partial \eta} \quad \text{at} \quad \eta = 0 \ \ (y \to 0) \tag{24}$$

where J is a dimensionless function of x (note also that in general $\delta = \delta(x)$)

$$J = \frac{k}{k_w}\left(\frac{U_\infty \delta^2}{\nu x}\right)^{1/2}. \tag{25}$$

The boundary condition of equation (24) accounts for the continuity of heat flux through the $y = 0$ surface, i.e. for the connection between convection above and conduction below this surface

$$k_w \frac{T-T_0}{\delta} = k \frac{\partial T}{\partial y} \quad \text{at } y = 0. \tag{26}$$

The limiting case of a wall with zero thickness, $J = 0$, is the same as Pohlhausen's [4] problem for heat transfer from an isothermal flat plate to a laminar boundary layer. The problem constructed here between equations (21) and (25) is a generalization to a wall with arbitrary thickness. In search of the optimal wall thickness function that minimizes the overall heat transfer rate per unit length q', we were unable to carry out the variational calculus using equations (21)–(25) in place of equation (9). Instead, we assumed a wall thickness shape that agrees *qualitatively* with the shape illustrated in Fig. 3

$$\frac{\delta}{\bar{\delta}} = 1 + b(\tfrac{1}{2} - \xi) \tag{27}$$

and then conducted a search for the minimum q'. In equation (27), b is a dimensionless wall taper parameter so that the thickness $\delta(\xi)$ decreases linearly from $(1+b/2)$ at the leading edge, to $(1-b/2)$ at $x = L$. Substituted in the J definition (25), the tapered shape (27) yields

$$J = \bar{J} \frac{1 + b(\tfrac{1}{2} - \xi)}{\xi^{1/2}} \tag{28}$$

in which $\bar{J}$ is a dimensionless constant

$$\bar{J} = \frac{k}{k_w} \frac{\bar{\delta}}{L} Re_L^{1/2}. \tag{29}$$

The $\bar{J}$ constant accounts for the fixed length and material inventory of the wall $(L, \bar{\delta})$, as a substitute for the constraint of equation (3) used in the first part of this study.

The total heat transfer rate

$$q' = \int_0^L k \left(\frac{\partial T}{\partial y} \right)_{y=0} dx$$

$$= k \Delta T Re_L^{1/2} \int_0^1 \left(\frac{\partial \theta}{\partial \eta} \right)_{\eta=0} \xi^{-1/2} \, d\xi \tag{30}$$

was determined numerically, by solving equations (21)–(24) for a fixed pair $(Pr, \bar{J})$. To start with, the Blasius function $f(\eta)$ was determined based on the one-time shooting method (Goldstein [5], Rosenhead [6], Van Dyke [7]; see also Bejan [3], pp. 61–62). Equation (21) was solved next by the centred finite difference method with second order accuracy. The uniform step $\Delta\eta = 0.01$ was chosen based on the information presented in Fig. 6, in which the local Nusselt number

$$Nu_x = \frac{hx}{k} = \frac{x}{\Delta T} \left(\frac{\partial T}{\partial y} \right)_{y=0} \tag{31}$$

was compared with Pohlhausen's asymptote for the $Pr \gg 1$ limit, $Nu_x = 0.332\, Pr^{1/3}\, Re_x^{1/2}$. Figure 6 shows

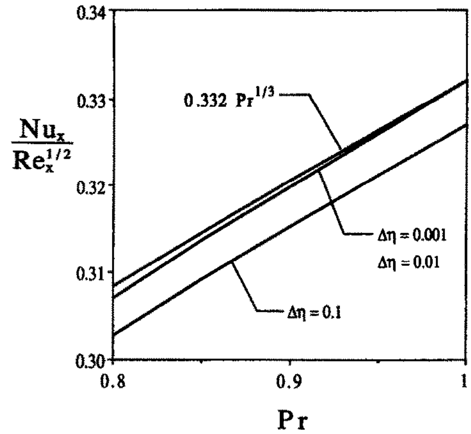

FIG. 6. The effect of grid size on the accuracy of the numerical solution ($\bar{J} = 0$).

that the $\Delta\eta = 0.01$ curve cannot be distinguished from the $\Delta\eta = 0.001$ curve, i.e. that the $\Delta\eta = 0.01$ grid is already fine enough.

The heat transfer results are reported in relative terms in Figs. 7 and 8. On the ordinate, the numerator

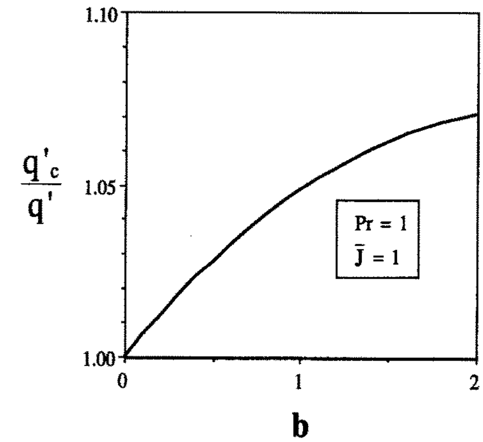

FIG. 7. The effect of the taper parameter b on the heat transfer rate through a laminar boundary layer adjacent to a wall of variable thickness, equation (27).

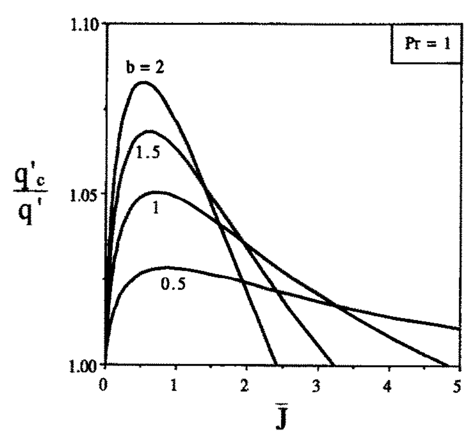

FIG. 8. The effect of the physical constraint parameter $\bar{J}$ on the total heat transfer rate through a laminar boundary layer adjacent to a wall of variable thickness, equation (27).

Table 2. The location of the maximum of the ratio q'_c/q' reported in Fig. 8 ($Pr = 1$)

b	$\bar{J}_{max}$	(q'_c/q'_{max})
0.5	0.88	1.028
1.0	0.76	1.050
1.5	0.58	1.068
2.0	0.50	1.083

q'_c represents the total heat transfer rate (per unit length) when the wall thickness is constant ($\delta = \bar{\delta}$ or $b = 0$). The denominator q' is the heat transfer rate (per unit length) through the wall with variable thickness, $0 < b \leqslant 2$. Note that when $b = 2$ the wall thickness is zero at the trailing edge ($x = L$).

Figure 7 shows that the ratio q'_c/q' increases steadily as the taper parameter b increases. This finding confirms the trend discovered in Fig. 2 (the solid lines), in which the taper becomes more pronounced as Bi decreases. The magnitude of the q'_c/q' ratio is comparable with the values plotted in Fig. 2.

The effect of changing the physical constraint parameter $\bar{J}$ is illustrated in Fig. 8. We see here that when the taper is fixed (for example, triangular wall profile, $b = 2$) the insulation effect reaches a maximum at a certain $\bar{J}$ value of order 1. The numerical details of the maxima of Fig. 8 are listed in Table 2. At the maximum (where $\bar{J}$ of equation (29) is a certain constant, $\bar{J}_{max}$), the average wall thickness $\bar{\delta}$ is proportional to $L^{1/2}$.

The actual heat transfer rate q' can be estimated by combining the relative information of Figs. 7 and 8 with the q'_c data plotted in Fig. 9. The latter is the final chart for the total heat transfer rate through a wall of constant thickness in contact with a forced convection laminar boundary layer. The chart shows how the $\bar{J}$ constant differentiates between situations in which the overall thermal resistance is dominated by the wall ($\bar{J} \gg 1$), and situations where the boundary layer has the greater of the two thermal resistances ($\bar{J} \ll 1$).

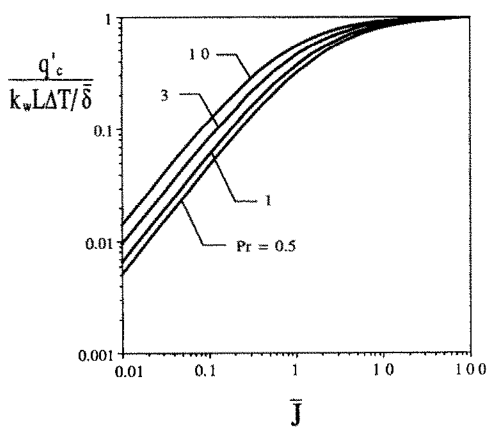

FIG. 9. The total heat transfer rate through a wall of constant thickness, lined by a forced convection laminar boundary layer.

By using equation (29) and the classical Pohlhausen solution, it is easy to verify that the asymptotes of each of the curves plotted in Fig. 9 are

$$\frac{q'_c}{k_w L\Delta T/\bar{\delta}} = \begin{cases} 1, & \text{if } \bar{J} \gg 1 \\ 0.664\, Pr^{1/3}\bar{J} & \text{if } \bar{J} \ll 1, \text{ and } Pr \geqslant 0.5. \end{cases} \quad (32)$$

CONCLUSIONS

In this paper we analyzed the conjugate heat transfer through a variable-thickness wall lined by a convective boundary layer. In the first part, the analysis was simplified by the assumption that the x-dependence of the wall–fluid heat transfer coefficient is known. Variational calculus showed that the total heat transfer rate is minimized when the wall thickness decreases in an optimal manner in the direction of flow.

In the second part of the paper, the complete problem of laminar boundary layer convection coupled with wall conduction was solved numerically. It was assumed that the wall thickness varies linearly with x; however, the taper of the wall profile (b) could change. The heat transfer rate decreased at larger values of b, i.e. as more of the wall material was shifted toward the leading edge. This conclusion agreed qualitatively with the one reached via variational calculus.

Finally, with reference to Fig. 2 it is important to recognize that the range $Bi < 1$ corresponds to a wall with an internal resistance smaller than that of the boundary layer. The figure showed that at the opposite end ($Bi > 1$) the use of a variable-thickness wall does not result in a significant reduction in the total heat transfer rate. In other words, when the wall internal resistance dominates the overall resistance from $T_0+\Delta T$ to T_0, the constant-thickness wall is practically as good as the wall with optimal thickness profile.

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REFERENCES

1. A. Bejan, *Advanced Engineering Thermodynamics*, pp. 722–723. Wiley, New York (1988).
2. W. M. Kays and M. E. Crawford, *Convective Heat and Mass Transfer*, p. 136. McGraw-Hill, New York (1980).
3. A. Bejan, *Convection Heat Transfer*, p. 50. Wiley, New York (1984).
4. E. Pohlhausen, Der Wärmeaustausch zwischen Korpern und Flüssigkeiten mit kleiner Reibung und kleiner Wärmeleitung, *Z. Angew. Math. Mech.* **1**, 115–121 (1921).
5. S. Goldstein (Editor), *Modern Developments in Fluid Dynamics*, p. 135. Oxford University Press, London (1938).
6. L. Rosenhead, *Laminar Boundary Layers*, p. 223. Oxford University Press, London (1963).
7. M. Van Dyke, *Perturbation Methods in Fluid Mechanics*, p. 131. Parabolic Press, Stanford, CA (1975).

EPAISSEUR OPTIMALE D'UNE PAROI AVEC CONVECTION SUR UNE FACE

Résumé—On considère le transfert thermique conjugué à travers une paroi avec épaisseur non uniforme et une condition limite sur une face. Dans une première partie, un calcul variationnel montre que le transfert thermique total est minimisé quand l'épaisseur de la paroi décroît d'une façon optimale dans la direction de l'écoulement. Les réductions du transfert total de chaleur sont significatives lorsque le nombre de Biot est inférieur à 1. Dans la seconde partie de l'étude, le problème complet d'une couche limite de convection forcée laminaire couplée à la conduction à travers une paroi à épaisseur variable est résolu numériquement. On rapporte graphiquement des moyens de calcul du flux de transfert total thermique. On trouve aussi que ce flux décroît quand le profil de la paroi est tel que l'épaisseur diminue dans la direction de l'écoulement.

DIE OPTIMALE DICKE EINER WAND MIT EINSEITIGER KONVEKTION

Zusammenfassung—Die vorliegende Arbeit befaßt sich mit dem konjugierten Wärmetransport durch eine Wand ungleichförmiger Dicke und einseitiger Grenzschicht. Im ersten Teil zeigt eine Variationsrechnung, daß der Wärmedurchgang durch Wahl einer abnehmenden Wanddicke in Strömungsrichtung auf optimale Weise minimiert wird. Die Verringerung des Wärmedurchgangs wird für Biot-Zahlen kleiner als 1 signifikant. Im zweiten Teil der Untersuchung wird das vollständige Problem einer laminaren Grenzschicht bei erzwungener Konvektion in Verbindung mit der Wärmeleitung durch die Wand variabler Dicke numerisch gelöst. Es folgt eine grafische Darstellung des Verfahrens zur Berechnung des Wärmedurchgangs. Es zeigt sich wieder, daß der Wärmedurchgang für den Fall eines sich verjüngenden Wandprofils abnimmt, so daß die Wanddicke in Strömungsrichtung abnimmt.

ОПТИМАЛЬНАЯ ТОЛЩИНА СТЕНКИ ПРИ НАЛИЧИИ КОНВЕКЦИИ С ОДНОЙ СТОРОНЫ

Аннотация—Исследуется сопряженный теплоперенос в случае стенки неоднородной толщины с пограничным слоем на одной стороне. Проведенные в первой части работы вариационные расчеты показывают, что результирующая скорость теплопереноса минимизируется при оптимальном уменьшении толщины стенки в направлении течения. Снижение скорости теплопереноса является существенным, когда значение числа Био меньше 1. Во второй части исследования численно решается совместная задача пограничного слоя при ламинарной вынужденной конвекции и теплопроводности через стенку переменной толщины. Графически представлены методы расчета результирующей скорости теплопереноса. Найдено, что она снижается в случае конусообразного профиля стенки, когда толщина стенки уменьшается в направлении течения.